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Results on relatively prime domination polynomial of some graphs

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Abstract

Let G be a non–trivial graph. A set $S \subseteq V$ is said to be a relatively prime dominating set if it is a dominating set with at least two elements and for every pair of vertices u and v in S such that $(d(u), d(v)) = 1$. The minimum cardinality of a relatively prime dominating set is called the relatively prime domination number and it is denoted by $\gamma_{rpd}(G)$. The relatively prime domination polynomial of a graph G of order n is the polynomial

$$D_{rpd}(G,x) = \sum_{k=\gamma_{rpd}(G)}^{n} d_{rpd}(G,k)x^k,$$

where $d_{rpd}(G,k)$ is the number of relatively prime dominating sets of G of size k, and $\gamma_{rpd}(G)$ is the relatively prime domination number of G. In this paper, we compute this polynomial for graphs $P_n^{---}, K_{1,n}^v, C_n^v, \bar{K}_{m,n}^V$ and B_n^v.

Keywords

Dominating polynomial, relatively prime dominating polynomial, relatively prime dominating polynomial roots.

AMS Subject Classification

05C69, 11B83.

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Contents

1. Introduction

By a graph $G = (V,E)$ we mean a finite undirected graph without loops and multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretical terms, we refer to Harary [3] and for terms related to domination we refer to Haynes [4].

A subset S of V is said to be a dominating set in G if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G.

Berge and Ore [2, 10] formulated the concept of domination in graphs. It was further extended to define many other domination related parameters in graphs. Let G be a non–trivial graph. A set $S \subseteq V$ is said to be a relatively prime dominating set if it is a dominating set and for every pair of vertices u and v in S such that

$$(d(u), d(v)) = 1.$$

The minimum cardinality of a relatively prime dominating set is called the relatively prime domination number and it is denoted by $\gamma_{rpd}(G)$ [6]. Switching in graphs was introduced by Lint and Seidel [9].

For a finite undirected graph $G(V,E)$ and a subset $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^{\sigma}(V,E')$ which is obtained from G by removing all edges between σ and its complement $V - \sigma$ and adding as edges all non-edges between σ and $V - \sigma$.

For $\sigma = \{v\}$, we write G^v instead of $G^{\{v\}}$ and the corresponding switching is called as vertex switching [5]. For more details about the basic definitionswe refer to Harrary [3].

Graph polynomials are powerful and well-developed tools to express graph parameters. SaeidAlikhani and Peng, Y. H. [1], have introduced the Domination polynomial of a graph.

The Domination polynomial of a graph G of order n is the polynomial

$$D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i,$$

where $d(G,i)$ is the number of dominating sets of G of size i, and $\gamma(G)$ is the domination number of G. This motivated us to introduce the relatively prime domination polynomial of a graph. In this paper, we find the relatively prime domination polynomial of some graphs.

2. Definition and Examples

Definition 2.1. *Let $G = (V,E)$ be a graph of order n with relatively prime domination number $\gamma_{rpd}(G)$. The relatively prime domination polynomial of G is,*

$$D_{rpd}(G,x) = \sum_{k=\gamma_{rpd}(G)}^{n} d_{rpd}(G,k)x^k,$$

where $d_{rpd}(G,k)$ is the number of relatively prime dominating sets of G of size k and $\gamma_{rpd}(G)$ is the relatively prime domination number of G. The roots of the polynomial $D_{rpd}(G,k)$ are called the relatively prime dominating roots of G.

Example 2.2. *Consider the graph G given in figure 2. 1. Clearly $\gamma_{rpd}(G) = 2$ and there are only two minimum relatively prime dominating sets of size 2, namely $\{v_2,v_4\}$ and $\{v_3,v_5\}$, three relatively prime dominating sets of size 3, namely $\{v_1,v_4,v_5\},\{v_1,v_3,v_5\}$ and $\{v_1,v_2,v_4\}$ and two relatively prime dominating sets of size 4, namely $\{v_1,v_2,v_4,v_5\}$ and $\{v_1,v_3,v_4,v_5\}$. Hence*

$$D_{rpd}(G,x) = 2x^2 + 3x^3 + 2x^4.$$

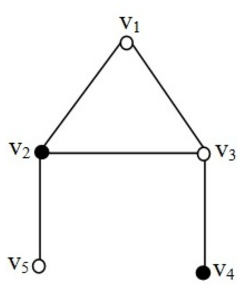

Fig. 2.1. G

Example 2.3. *Consider the graph $G = 2K_2$ given in figure 2.2. Clearly $\gamma_{rpd}(G) = 2$ and there are only four minimum relatively prime dominating sets of size 2, namely $\{v_1,v_3\},\{v_1,v_4\},\{v_2,v_3\}$ and $\{v_2,v_4\}$, four relatively prime dominating sets of size 3, namely $\{v_1,v_2,v_3\},\{v_1,v_2,v_4\},\{v_1,v_3,v_4\}$ and $\{v_2,v_3,v_4\}$ and one relatively prime dominating set of size 4 which is $\{v_1,v_2,v_3,v_4\}$. Hence*

$$D_{rpd}(G,x) = 4x^2 + 4x^3 + x^4 = x^2(4+4x+x^2).$$

Obviously, there are two relatively prime dominating roots of G which are 0 and –2.

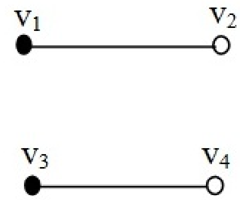

Fig. 2.2. $G = 2K_2$

Definition 2.4. *[11]Let $G = (V,E)$ be a graph and let x,y,z be three variables taking values $+$ or $-$. The* ***transformation graph*** *G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for $\alpha,\beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds:*

(i) *$\alpha,\beta \in V(G)$. α and β are adjacent in G if $x = +$; α and β are not adjacent in G if $x = -$.*

(ii) *$\alpha,\beta \in E(G)$. α and β are adjacent in G if $y = +$; α and β are not adjacent in G if $y = -$.*

(iii) *$\alpha \in V(G),\beta \in E(G)$. α and β are incident in G if $z = +$; α and β are not incident in G if $z = -$.*

Thus, we may obtain eight kinds of transformation graphs, in which G^{+++} is the total graph of G, and G^{---} is its complement. Also, G^{--+},G^{-+-} and G^{-++} are the complements of G^{++-},G^{+-+} and G^{+--}, respectively.

Example 2.5. *The graph $G = C_4$ and G^{---} are given in figure 2.3.*

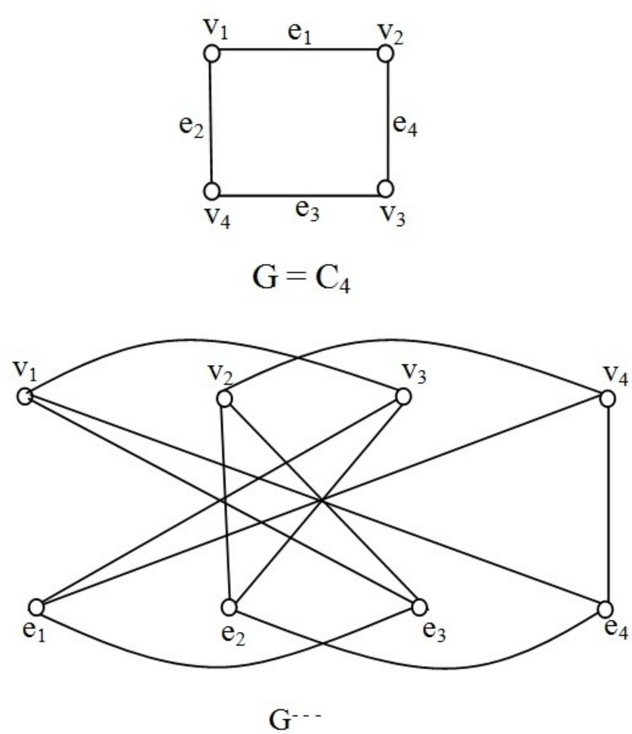

Fig. 2.3.

We recall the following theorems for future study.

Theorem 2.6. *[6] For a complete bipartite graph*

$$K_{m,n}, \gamma_{rpd}(K_{m,n}) = 2$$

if and only if $(m,n) = 1$.

Theorem 2.7. *[7] If* $G_1 \cong G_2$, *then*

$$D_{rpd}(G_1,x) = D_{rpd}(G_2,x).$$

Theorem 2.8. *[7] For* $m,n \geq 2$,

$$D_{rpd}(K_{m,n},x) = mnx^2$$

if $(m,n) = 1$.

Theorem 2.9. *[8]*

$$\gamma_{rpd}(C_n^v) = \begin{cases} 2 & \textit{for} \quad 3 \leq n \leq 6 \\ 3 & \textit{for} \quad n \geq 7 \end{cases}$$

Theorem 2.10. *[7] Let* $G = K_m \cup K_n$ *where* $m,n \geq 2$. *Then,*

$$D_{rpd}(G,x) = mnx^2$$

if $(m-1,n-1) = 1$.

Theorem 2.11. *[8] Let G be the book graph* $B_n, n \geq 2$ *and v be any vertex of G.*

i) *If* $d_G(v) = 2$ *and* $n \equiv 3(\mod 10)$, *then* $\gamma_{rpd}(G^v) = 3$.

ii) *If* $d_G(v) = 2$ *and* $n \neq 3(\mod 10)$, *then* $\gamma_{rpd}(G^v) = 2$.

iii) *If* $d_G(v) = n$, *then* $\gamma_{rpd}(G^v) = n$.

3. Main Results

In this section, we compute $D_{rpd}(G,x)$ where G is $P_n^{---} (n \geq 4), K_{1,n}^v, C_n^v, \bar{K}_{m,n}^V$ and B_n^v.

Result 3.1. *Let* $G = P_3^{---}$. *Then* $D_{rpd}(G,x) = x^3$.

Theorem 3.2. *Let* $G = P_n^{---}, n \geq 4$. *Then* $D_{rpd}(G,x) = 2x^2$.

Proof. Let P_n be $v_1e_1v_2e_2...e_{n-1}v_n$. Let G be the transformation graph P_n^{---}. Then

$$V(P_n^{---}) = \{v_1,v_2,...,v_n,e_1,e_2,e_3,...,e_{n-1}\}.$$

Clearly, there are only two minimal relatively prime dominating sets of size 2, namely $\{v_1,e_{n-1}\}$ and $\{v_n,e_1\}$. This implies that $d_{rpd}(G,2) = 2$. Any relatively prime dominating set with more than 2 vertices must contain at least two vertices of same degree. Therefore,

$$d_{rpd}(G,3) = d_{rpd}(G,4) = ... = 0.$$

Hence

$$D_{rpd}(G,x) = 2x^2.$$

□

Theorem 3.3. *For the star* $K_{1,n}$ *where* $n \geq 2$ *is even,*

$$D_{rpd}(K_{1,n}^v,x) = 2(n–1)x^2,$$

if v is an end vertex of $K_{1,n}$.

Proof. Let u be the centre and v be an end vertex of $K_{1,n}$. In $K_{1,n}^v$ the vertex u and v are adjacent to all other vertices.

Hence

$$K_{1,n}^v \cong K_{2,n-1}.$$

By Theorem 2.6,

$$\gamma_{rpd}(K_{2,n-1}) = 2$$

if and only if $(2,n–1) = 1$. This implies that $n–1 \neq 2r$.

Therefore, $n \neq 2r+1$ and hence n is even.

Clearly $(2,n–1) = 1$. By Theorems 2.7 and 2.8,

$$D_{rpd}(K_{1,n}^v,x) = D_{rpd}(K_{2,n–1},x) = 2(n–1)x^2.$$

□

Result 3.4.

$$D_{rpd}(C_3^v,x) = 2x^2 + x^3.$$

Result 3.5.

$$D_{rpd}(C_4^v,x) = 3x^2 + 3x^3 + x^4.$$

Result 3.6.

$$D_{rpd}(C_5^v,x) = 2x^2 + 5x^3 + 2x^4.$$

Result 3.7.

$$D_{rpd}(C_6^v,x) = 2x^2 + 3x^3.$$

Theorem 3.8. *For* $n \geq 7$,

$$D_{rpd}(C_n^v,x) = \begin{cases} 3x^3 + (n-3)x^4 & \textit{if} \quad n \neq 3+3r, r > 1 \\ x^3 \quad \textit{if} \quad n = 3+3r \end{cases}$$

Proof. Let G be the graph C_n^v and let $v_1v_2...v_nv_1$ be the cycle C_n. By Theorem 2.9, $\gamma_{rpd}(C_n^v) = 3$ for $n \geq 7$. Without loss of generality, let v be v_1. Now, $d_G(v_1) = n–3, d_G(v_2) = d_G(v_n) = 1$ and $d_G(v_i) = 3, 3 \leq i \leq n–1$. We consider the following two cases.

Case 1. $n \neq 3+3r, r > 1$

Then $d_G(v_1) = n-3 \neq 3r$. Hence there are only three minimum relatively prime dominating sets of size 3, namely $\{v_1,v_2,v_n\}, \{v_1,v_3,v_n\}$ and $\{v_1,v_2,v_{n-1}\}$. Hence

$$d_{rpd}(G,3) = 3.$$

Also, there are $n-3$ relatively prime dominating sets of size 4, namely

$\{v_1,v_2,v_3,v_n\}, \{v_1,v_2,v_4,v_n\},...,\{v_1,v_2,v_{n-1},v_n\}$.

Therefore,

$$d_{rpd}(G,4) = n-3.$$

Any dominating set that contains more than four vertices must contain at least two vertices of same degree 3 and hence there is no relatively prime dominating set exists with more than four vertices. This implies that

$$d_{rpd}(G,5) = d_{rpd}(G,6) = ... = 0.$$

Hence

$$D_{rpd}(G,x) = d_{rpd}(G,3)x^3 + d_{rpd}(G,4)x^4 = 3x^3 + (n\text{–}3)x^4.$$

Case 2. $n = 3+3r, r > 1$

Here $d_G(v_1) = n-3 = 3r$, which is a multiple of 3. Clearly, there is only one minimum relatively prime dominating set of size 3, namely $\{v_1, v_2, v_n\}$ and hence $d_{rpd}(G,3) = 1$. Any dominating set that contains more than three vertices must contain at least two vertices of same degree and hence there is no relatively prime dominating set exists with more than three vertices. This implies that

$$d_{rpd}(G,4) = d_{rpd}(G,5) = ... = 0.$$

Hence,

$$D_{rpd}(G,x) = x^3.$$

The theorem follows from cases 1 and 2. □

Theorem 3.9. *For $n \geq 2$, $D_{rpd}\left(\bar{K}^V_{m,n}, x\right) = (m-1)(n+1)x^2$ or $(m+1)(n-1)x^2$, according as $v \in V_1$ and $(m-1, n+1) = 1$ or $v \in V_2$ and $(m+1, n-1) = 1$ where (V_1, V_2) is a bipartition of the vertex set of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$.*

Proof. Let (V_1, V_2) be the bipartition of the vertex set of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Clearly $\bar{K}_{m,n} = K_m \cup K_n$. Now, $\bar{K}^V_{m,n}$ is either $K_{m-1} \cup K_{n-1}$ or $K_{m+1} \cup K_{n-1}$ according as $v \in V_1$ or $v \in V_2$. By Theorem 2.10,

$$D_{rpd}\left(\bar{K}^V_{m,n}, x\right) = (m-1)(n+1)x^2 or (m+1)(n-1)x^2,$$

according as $(m-1, n+1) = 1$ or $(m+1, n-1) = 1$.

This completes the proof. □

Theorem 3.10. *Let G be the Book graph $B_n, n \geq 2$ and $d_G(v) = n$. Then,*

$$D_{rpd}(G^v, x) = \begin{cases} nx^n + 3nx^{n+1} + 2nx^{n+2} & \textit{if} \quad n \neq 0(mod3) \\ nx^n + nx^{n+1} & \textit{if} \quad n = 0(mod3) \end{cases}$$

Proof. Let $v_0, v_1, ..., v_n$ and $u_0, u_1, ..., u_n$ be the two copies of star $K_{1,n}$ with central vertices v_0 and u_0 respectively. Join u_i with v_i for all $i, 1 \leq i \leq n$. The resultant graph G is B_n with vertex set

$$V(G) = \{v_0, u_0, v_i, u_i / 1 \leq i \leq n\}$$

and edge set

$$E(G) = \{u_0v_0, u_iv_i, v_0v_i, u_0u_i / 1 \leq i \leq n\}.$$

Then G has $2n+2$ vertices and $3n+1$ edges and $d_G(v) = 2$ if $v \in \{u_i, v_i / 1 \leq i \leq n\}$ and $d_G(v) = n$ if $v \in \{u_0, v_0\}$. Let $A = \{u_1, u_2, ..., u_n\}$ and $B = \{v_1, v_2, ..., v_n\}$. We consider the following two cases on n.

Case 1. $n \neq 0(mod3)$

By Theorem 2.11, $\gamma_{rpd}(G^v) = n$ if $d_G(v) = n$. A minimal relatively prime dominating set of size n is obtained by selecting a vertex from A and n–1 vertices from B. This can be done in n ways.

Therefore,

$$d_{rpd}(G,n) = n.$$

A relatively prime dominating set of size $n+1$ is obtained by selecting either the vertex set B and a vertex of A which can be done in n ways or a vertex u_i of A, $n-1$ vertices from $B - \{v_i\}$ and the vertex v_0 which can be done in n ways or a vertex u_i of A, $n-1$ vertices from $B - \{v_i\}$ and the vertex u_0 which can be done in n ways.

Therefore,

$$d_{rpd}(G, n+1) = 3n.$$

A relatively prime dominating set of size $n+2$ is obtained by selecting either the vertex set B, a vertex from A and the vertex v_0, which can be done in n ways or the vertex set B, a vertex from A and the vertex u_0, which can be done in n ways.

Therefore,

$$d_{rpd}(G, n+2) = 2n.$$

Any relatively prime dominating set of size more than $n+2$ vertices must contain at least two vertices of same degree.

Therefore,

$$d_{rpd}(G, n+3) = ... = 0.$$

Hence,

$$D_{rpd}(G^v, x) = d_{rpd}(G,n)x^n + d_{rpd}(G,n+1)x^{n+1}$$

$$+ d_{rpd}(G,n+2)x^{n+2} = nx^n + 3nx^{n+1} + 2nx^{n+2}.$$

Case 2. $n \equiv 0(mod3)$

A minimal relatively prime dominating set of size n is obtained by selecting a vertex u_i from A and $n-1$ vertices from $B - \{v_i\}$. This can be done in n ways. Therefore, $d_{rpd}(G,n) = n$. A relatively prime dominating set of size $n+1$ is obtained by selecting the vertex set B and a vertex of A, this can be done in n ways. Therefore, $d_{rpd}(G, n+1) = n$. Any relatively prime dominating set of size more than $n+1$ vertices must contain at least two vertices of same degree. Therefore,

$$D_{rpd}(G^v, x) = d_{rpd}(G,n)x^n + d_{rpd}(G,n+1)x^{n+1} = nx^n + nx^{n+1}.$$

The theorem follows from cases 1 and 2. □

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